Correlations of eigenvectors for non-Hermitian random-matrix models

Romuald A. Janik,[1,2] Wolfgang Nörenberg,[3] Maciej A. Nowak,[2,3] Gábor Papp,[4] and Ismail Zahed[5]
[1]*Service de Physique Théorique, CEA Saclay, F-91191 Gif-Sur-Yvette, France*
[2]*Marian Smoluchowski Institute of Physics, Jagellonian University, 30-059 Krakow, Poland*
[3]*GSI, Planckstrasse 1, D-64291 Darmstadt, Germany*
[4]*CNR, Department of Physics, KSU, Kent, Ohio 44242*
and Institute for Theoretical Physics, Eötvös University, Budapest, Hungary
[5]*Department of Physics and Astronomy, SUNY at Stony Brook, Stony Brook, New York 11794*
(Received 24 February 1999)

We establish a general relation between the diagonal correlator of eigenvectors and the spectral Green's function for non-Hermitian random-matrix models in the large-N limit. We apply this result to a number of non-Hermitian random-matrix models and show that the outcome is in good agreement with numerical results. [S1063-651X(99)09109-6]

PACS number(s): 05.40.−a, 05.30.−d, 03.65.Nk, 24.60.−k

I. INTRODUCTION

Recently, a number of new results for non-Hermitian random-matrix (NHRM) ensembles were obtained (see, e.g., [1–5] and references therein), reflecting on the rapidly growing interest in properties of NHRM in several areas of physics. In a recent paper [6], Chalker and Mehlig have pointed out the existence of remarkable correlations between left and right eigenvectors associated with pairs of eigenvalues lying close in the complex plane. Such effects do not exist for Hermitian (more generally normal) random-matrix models, since in this case the left and right eigenvectors coincide. Some observables related to the eigenvector properties in non-Hermitian random-matrix models have been introduced and studied numerically in [7]. Chalker and Mehlig [6] obtained analytical formulas (in the large-N limit, where N is the size of the NHRM) for correlations between left and right eigenvectors in the case of Ginibre's ensemble [8]. However, an efficient calculational scheme for the simplest (and perhaps physically more transparent) one-point function $O(z)$ was lacking. In this paper we prove a simple relation stating that the correlator between left and right eigenvectors corresponding to the same eigenvalue is exactly equal, in the large-N limit, to the square of the off-diagonal one-point Green's function [4] for non-Hermitian eigenvalues. The latter is readily calculable for a wide variety of NHRM ensembles. We illustrate our observation in a number of NHRM models and show that it agrees with numerical calculations.

The present results can be used to study the interplay between reorganization of the left and right eigenvectors and structural changes in the complex spectrum. In the simplest non-Hermitian model—Ginibre's ensemble (or generalizations thereof [9,10])—the density of complex eigenvalues is constant, filling uniformly the circle (ellipse) in the complex plane. This model and its variants do not have external parameters, which could induce structural changes (''phase changes'') in the average eigenvalue distribution, e.g., the changes from simply to multiply connected domain of eigenvalues. In order to study these more complex phenomena one has to consider, e.g., the model for open chaotic scattering, where at large couplings (strong dissipation) the eigenvalue distribution splits into two disconnected islands, reflecting on the separation of time scales [11,12]. One island corresponds to short-lived resonances, while the other corresponds to long-lived (almost classical) trapped states [13]. It is known in this case that the reorganization of the eigenvalues is followed by some reorganization of the eigenvectors [7]. In particular, the norm of the right states is sensitive to the distribution of resonances [7].

In Sec. II, we introduce the notations and summarize the main results for the eigenvector correlators in the case of Ginibre's ensemble [8], as established recently by Chalker and Mehlig [6]. In Sec. III, we present our main result and apply it to several NHRM models with direct comparison to numerical results. Our analysis relies on novel techniques for NHRM models discussed by some of us [4]. A summary of our results is given in Sec. IV, and a number of technical details can be found in the Appendixes.

II. GINIBRE'S MATRIX MODEL

Ginibre [8] has introduced a Gaussian ensemble of general complex matrices $N\times N$, i.e., matrices distributed with the probability

$$P(\mathcal{M})d\mathcal{M}\sim \exp(-N\ \mathrm{Tr}\ \mathcal{M}\ \mathcal{M}^{\dagger})d\mathcal{M} \tag{1}$$

giving nonvanishing cumulants $\langle \mathcal{M}_{ab}\overline{\mathcal{M}}_{ab}\rangle = 1/N$. The eigenvalues are uniformly distributed on a unit disk centered at the origin of the complex plane. In Appendix B, we provide a short derivation of this result and others using (matrix-valued) Blue's functions [4,5]. Since the matrices are complex (non-Hermitian), there exists a biorthogonal set of right (R) and left (L) eigenvectors, so that

$$\mathcal{M}=\sum_a \lambda_a |R_a\rangle\langle L_a|,$$
$$\mathcal{M}^{\dagger}=\sum_b \overline{\lambda}_b |L_b\rangle\langle R_b|, \tag{2}$$

where $\langle L_a|R_b\rangle=\delta_{ab}$. Chalker and Mehlig [6] have studied the following eigenvector correlators:

$$O(z)=\left\langle \frac{1}{N}\sum_a O_{aa}\delta(z-\lambda_a)\right\rangle,$$
$$O(z,w)=\left\langle \frac{1}{N}\sum_{a\neq b} O_{ab}\delta(z-\lambda_a)\delta(w-\lambda_b)\right\rangle, \tag{3}$$

where $O_{ab}=\langle L_a|L_b\rangle\langle R_b|R_a\rangle$. The main results of their paper are exact expressions for $O(z,w)$ and for $O(z)$ in the case of Ginibre's ensemble. For $N\gg 1, |z-w|\neq 0$, where z,w are lying within the unit circle

$$O(z,w)=-\frac{1}{\pi^2}\frac{1-z\bar{w}}{|z-w|^4}. \tag{4}$$

For $|z-w|\to 0$,

$$O(z,w)_{\rm micro}=-\frac{N^2}{\pi^2}\frac{1-|Z|^2}{|\omega|^4}[1-(1+|\omega|^2)\exp(-|\omega|^2)], \tag{5}$$

where $Z=(z+w)/2$ and $\omega=\sqrt{N}(z-w)$. The diagonal correlator reads

$$O(z)=\frac{N}{\pi}(1-|z|^2). \tag{6}$$

Whereas the calculation of the ''wide'' eigenvector correlators $O(z,w)$ [like Eq. (4)] in many cases of NHRM is straightforward, using, e.g., non-Hermitian diagrammatics [4], the calculation of the ''close'' eigenvector correlators $O(z-w)_{\rm micro}$ and the diagonal correlators $O(z)$ is technically involved. In practice, it seems that explicit calculations in the microscopic limit are possible only in the cases when the spectrum of NHRM possesses an azimuthal (rotational) symmetry. In the next section we provide a general formula for the diagonal eigenvector correlator $O(z)$ in terms of the spectral one-point Green's function.

III. FORMULA AND RESULTS

The main result of this paper reads

$$O(z)=-\frac{N}{\pi}\mathcal{G}_{qq}\bar{\mathcal{G}}_{qq}. \tag{7}$$

Here $\mathcal{G}_{qq}$ and $\mathcal{G}_{qq}$ are off-diagonal elements of the generalized (2×2) spectral Green's function $\mathcal{G}$ [4],

$$\mathcal{G}=\begin{pmatrix}\mathcal{G}_{qq} & \mathcal{G}_{q\bar{q}}\\ \mathcal{G}_{\bar{q}q} & \mathcal{G}_{\bar{q}\bar{q}}\end{pmatrix}. \tag{8}$$

The elements $\mathcal{G}_{ab}$ are defined as traces $\mathcal{G}_{ab}=(1/N)\mathrm{Tr}_N\hat{\mathcal{G}}_{ab}$ of the $N\times N$ blocks of the generalized resolvent [4],

$$\hat{\mathcal{G}}=\begin{pmatrix}\hat{\mathcal{G}}_{qq} & \hat{\mathcal{G}}_{q\bar{q}}\\ \hat{\mathcal{G}}_{\bar{q}q} & \hat{\mathcal{G}}_{\bar{q}\bar{q}}\end{pmatrix}=\left\langle\begin{pmatrix} z-\mathcal{M} & i\epsilon\mathbf{1}_N\\ i\epsilon\mathbf{1}_N & \bar{z}-\mathcal{M}^\dagger\end{pmatrix}^{-1}\right\rangle, \tag{9}$$

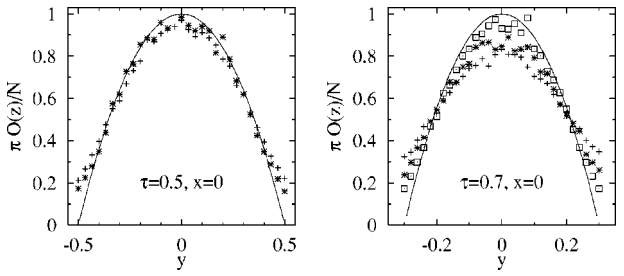

FIG. 1. Eigenvector correlator $O(z=0+iy)$, generated from an elliptic ensemble of 5×10^5 matrices of size 20 (crosses), an ensemble of 10^5 matrices of size 40 (stars) and size 100 (boxes) versus the analytical correlator $-\mathcal{G}_{qq}\mathcal{G}_{q\bar{q}}$, for $\tau=0.5$ (left) and $\tau=0.7$ (right).

where $\mathbf{1}_N$ is the N-dimensional identity matrix and $\langle\,\rangle$ denotes the averaging over the pertinent ensemble of random matrices $\mathcal{M}$. In comparison to the original work [4], we have chosen here purely imaginary infinitesimal values in the off-diagonal block. This way guarantees that the mathematical operations performed in the proof of Eq. (7) are well-defined. The result (7) follows from the more general formula (A11) and the assumption that no subtleties arise in the factorization theorem when interchanging the limits $N\to\infty$ and $\epsilon\to 0$ (cf. Appendix A).

The spectral density follows from Gauss' law [10],

$$\nu(z,\bar{z})=\frac{1}{\pi}\partial_{\bar{z}}\mathcal{G}_{qq}(z,\bar{z}), \tag{10}$$

which is the distribution of eigenvalues of $\mathcal{M}$. For Hermitian $\mathcal{M}$, Eq. (10) can be nonzero only on the real axis. As $\epsilon\to 0$, the block structure decouples, and we are left with the original resolvent. For $z\to+i0$, the latter is just a measurement of the real eigenvalue distribution.

For non-Hermitian $\mathcal{M}$, as $\epsilon\to 0$, the block structure does not decouple, leading to a nonholomorphic resolvent for certain two-dimensional domains on the z plane. For more technical details we refer to the original work [4], or recent reviews [14,15]. Similar constructions have been proposed recently in [5,16].

The right-hand side of the relation (7) is usually given by a simple analytical formula. Technically, the most efficient way of calculating the off-diagonal components of the Green's functions is to use the generalized Blue's function technique [4,5]. In Appendix B we provide a pedagogical derivation of some of the results below; for others we refer to the original papers.

For Ginibre's ensemble we immediately get (cf. Appendix B)

$$O(z)_{\rm Ginibre}=-\frac{N}{\pi}(\sqrt{z\bar{z}-1})^2=\frac{N}{\pi}(1-|z|^2) \tag{11}$$

in agreement with Chalker and Mehlig [6]. For the elliptic ensemble [9,10], using the off-diagonal elements from Appendix B, the diagonal correlator reads

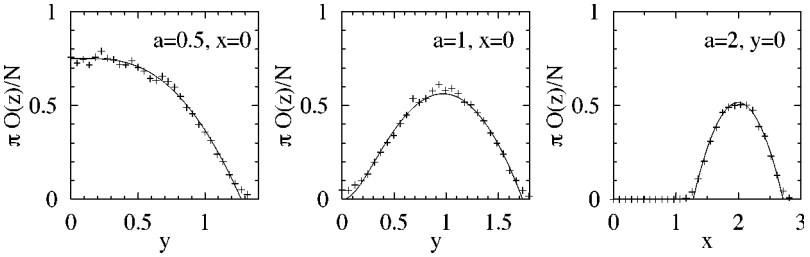

FIG. 2. ''Ginibre+deterministic'' eigenvector correlator $O(z=0+iy)$, generated from an ensemble of 10^5 matrices of size 60 (crosses) versus the analytical correlator $-\mathcal{G}_{qq}^{-}\mathcal{G}_{qq}^{-}$, at $a=0.5$ (left), $a=1$ (center), and the correlator $O(z=x+i0)$ at $a=2$ (right).

$$O(z)_{\text{elliptic}}=\frac{N}{\pi}\frac{1}{(1-\tau^2)^2}[(1-\tau^2)^2 -(1+\tau^2)|z|^2+2\tau\,\mathrm{Re}\,z^2]. \tag{12}$$

In Fig. 1, we present numerical results generated from computer simulation of eigenvectors for the elliptic ensemble with $\tau=0.5$ and $\tau=0.7$ and different size of the matrix, N, versus the analytical prediction (12). The results are satisfactory. The figure provides also a rough estimation of finite-size effects. We note that numerical simulations of eigenvector correlations are time consuming, hence the utility of the result (7), where the right-hand side is straightforward to calculate.

The models considered above have constant density of complex eigenvalues, and the domain of eigenvalues is simply connected (circle or ellipse). Below, we consider a toy model, where the domain could split into two disconnected domains at some critical value of the external parameter. Also the distribution of eigenvalues is nonuniform. The simplest non-Hermitian model of this kind is Ginibre's random Hamiltonian plus a two-level deterministic Hamiltonian, with $N/2$ levels a and $N/2$ levels $-a$. (To our knowledge, this model was first considered by Feinberg and Zee, using their hermitization method [5].) Using the addition law for the generalized Blue's function, we easily obtain all the components of the matrix-valued Green's function $\mathcal{G}$ (cf. Appendix B).

Using the relation (7), we predict

$$O(z)_{G+D}=-\frac{N}{\pi}\{|z|^2+a^2-\tfrac{1}{2}[1+\sqrt{1+4a^2(z+\overline{z})^2}]\}. \tag{13}$$

We note that the value $a=1$ is the critical one, when the single island of eigenvalues splits into two. Numerical simulations for this ensemble are shown in Fig. 2. The solid line is the analytical result, the crosses are the numerical results calculated for the line $z=0+iy$. The diagonal eigenvector correlators $O(z)$ *inside* the islands follow the distribution determined by the off-diagonal components for the spectral Green's function. We also note that the eigenvector correlator follows precisely the boundary shape of the eigenvalues. This is expected, since the condition $|\mathcal{G}_{qq}^{-}|=0$ determines [4] the regions in the complex plane separating the holomorphic and nonholomorphic components of the spectral Green's function (The shape of the eigenvalue domains for NHRM models can be inferred from associated Hermitian models using conformal mapping [4]. This points to yet another relationship between the eigenvalues and eigenvectors of Hermitian and non-Hermitian models.)

Finally, we consider the case of open chaotic scattering. As an illustration we choose the classical result of Haake, Sommers, and co-workers [12], based on the Mahaux-Weidenmüller [11] microscopic picture for nuclear reactions. In brief, the model is generically described by a non-Hermitian Hamiltonian of the form

$$H-igVV^{\dagger}, \tag{14}$$

where H is a random Gaussian (orthogonal) $N\times N$ matrix, while V is an $N\times M$ random matrix [12]. Here N is a number of discrete states and $M<N$ is a number of the continua. The model was solved [12] in the limit $N\to\infty$, $M\to\infty$, $\mathrm{m}\equiv M/N$ fixed. Using the results from Blue's function techniques (cf. Appendix B), we predict the analytical behavior for the correlator $O(z)$ for this model as

$$O(z)=\frac{N}{\pi}\left[\frac{1}{1-gy}-\frac{x^2}{4}-\frac{1}{4}\left(\frac{g}{1-gy}+\frac{m}{y}+\frac{1}{g}\right)^2\right] \tag{15}$$

with $z=x+iy$. In Fig. 3 we compare this result to a numerically generated ensemble of eigenvectors. We would like to note large finite-size effects at the edges of the islands.

Since in this model a change in the parameters g and m can generate structural changes in the distribution of eigenvalues, it is interesting to study whether the splitting of the ''islands'' is accompanied by some distinct behavior of the eigenvector correlators.

In Fig. 4 we plot the analytical result (15), normalized by the spectral density $\nu(z,\overline{z})$ (cf. Appendix B), as a function of x and y, where $z=x+iy$. For the case considered here (m

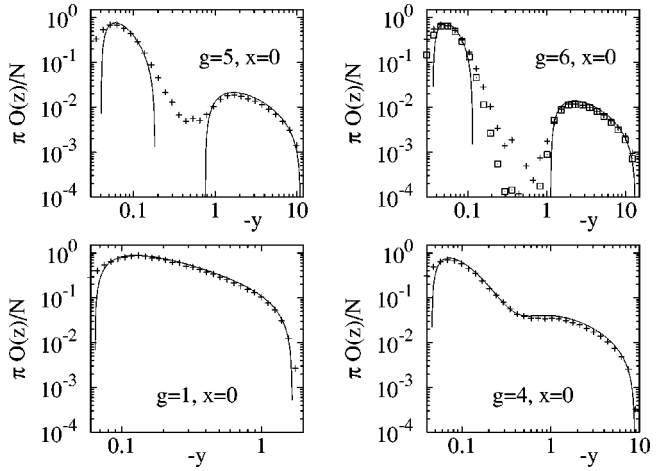

FIG. 3. Scattering model eigenvector correlator $O(z=0+iy)$ generated from an ensemble of 10^5 matrices of size 60 (crosses) and of size 120 (boxes) with $m=0.25$, versus the analytical correlator $-\mathcal{G}_{qq}^{-}\mathcal{G}_{qq}^{-}$, at different values of the coupling constant g.

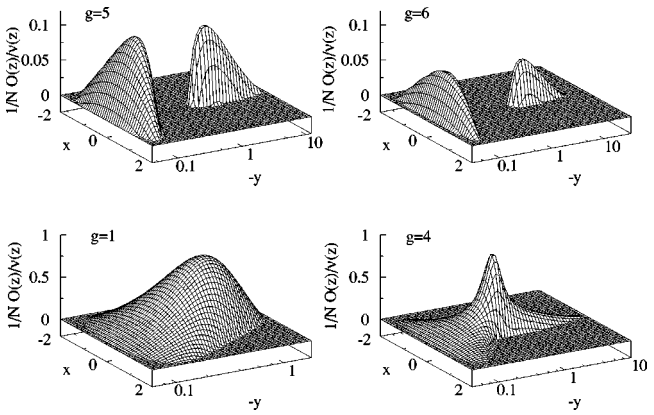

FIG. 4. Three-dimensional plots of the analytical results for the eigenvector correlator $O(z)$, normalized by the spectral density $\nu(z,\bar{z})$, vs x and y for different couplings g in the case of open chaotic scattering.

$=0.25$), the value of the critical coupling is $g=4.44$. [Generally, $g_{\rm crit}^2=(1-\sqrt[3]{m})^{-3}$ [12].] We observe a strong reorganization in the distribution of the average ''norm'' of eigenvectors in the vicinity of the critical coupling.

We have not taken up in this paper the issue of wide eigenvector correlators $O(z,w)$. For the cases considered here, these correlators are readily constructed using the NHRM diagrammatic Bethe-Salpeter equations [4], as noted by Chalker and Mehlig [6]. In most cases, however, the resulting final formulas are rather lengthy. We note that the knowledge of $O(z,w)$ in the regime $z-w\sim O(1)$ does not suffice to determine $O(z)$ through the sum rules [6] originating from the biorthogonality of the left/right eigenvectors. Hence the relevance of the present investigation.

IV. SUMMARY

We have established a relation between the diagonal correlator of eigenvectors and the off-diagonal elements of the one-point spectral Green's function for general ensembles of NHRM models. We have appplied this result to a number of NHRM models and checked its validity against numerically generated results. Our observation accounts for part of the eigenvector correlations established recently by a number of authors [6,7]. Our result generalizes to non-Hermitian ensembles with real, complex, or quaternionic components, as well as non-Hermitian ensembles with additional symmetry (e.g., chiral NHRM). In this last case, however, non-Hermitian diagrammatic techniques have to be used instead of the versatile method of Blue's functions.

Finally, we point out the possibility of relating the eigenvector correlators to the eigenvalue correlators in the microscopic limit. This issue and others will be discussed elsewhere.

ACKNOWLEDGMENTS

We would like to thank Evgueni Kolomeitsev for discussions. This work was supported in part by the U.S. DOE under Grant No. DE-FG-88ER40388 and DE-FG02-86ER40251, by the Polish Government Project (KBN) under Grant No. 2P03B00814, and by Hungarian Grant No. OTKA F026622.

APPENDIX A: PROOF OF EQ. (7)

We prove first the following representation for $O(z)$:

$$\lim_{\epsilon\to 0}\left\langle \mathrm{Tr}\frac{\epsilon}{(z-\mathcal{M})(\bar{z}-\mathcal{M}^\dagger)+\epsilon^2}\mathrm{Tr}\frac{\epsilon}{(\bar{z}-\mathcal{M}^\dagger)(z-\mathcal{M})+\epsilon^2}\right\rangle = \pi N O(z). \tag{A1}$$

Below we use $\mathrm{Tr}\ln A=\ln\det A$, and the fact that $\mathcal{M}$ can be diagonalized by a *nonunitary* transformation U, $U^{-1}\mathcal{M}U=\mathrm{diag}(\lambda_1,\ldots,\lambda_N)$. Introducing $V=U^{-1}U^{-1\dagger}$ we get

$$\begin{aligned}&\mathrm{Tr}\frac{\epsilon}{(z-\mathcal{M})(\bar{z}-\mathcal{M}^\dagger)+\epsilon^2}\\ &\quad=\frac{1}{2}\frac{d}{d\epsilon}\ln\det\left[(z-\lambda_i)V_{ik}(\bar{z}-\bar{\lambda}_k)+\epsilon^2 V_{ik}\right]\\ &\quad\equiv\tfrac{1}{2}\partial_\epsilon\ln\det[\cdots].\end{aligned} \tag{A2}$$

Using this representation, we note that $O(z)$ is zero in the limiting procedure, unless z is close to any (say λ_1) of the eigenvalues of $\mathcal{M}$. If this happens, we use the parametrization $z-\lambda_1=\epsilon u\exp(i\phi)$ with $u\sim O(1)$. Of course we get additional (similar) contributions when z is close to the other eigenvalues λ_i. For notational simplicity we will now consider only the case of λ_1, adding the remaining contributions in Eq. (A7).

Using Laplace's expansion for the first row of the determinant in Eq. (A2), we arrive in the $\epsilon\to 0$ limit,

$$\tfrac{1}{2}\partial_\epsilon\ln\det[\cdots]=\frac{1}{\epsilon}\frac{1}{1+u^2\dfrac{\det W_2}{V_{11}\det W_1}}, \tag{A3}$$

with W_1 being the matrix $(z-\lambda_i)V_{ik}(\bar{z}-\bar{\lambda}_k)$ without the first row and column and $(W_2)_{i+1,k+1}=(W_1)_{ik}$, $(W_2)_{1,k}=(\bar{z}-\bar{\lambda}_k)V_{1k}$, $(W_2)_{i,1}=(z-\lambda_i)V_{i1}$, and $(W_2)_{11}=V_{11}$.

From Laplace's expansion

$$\begin{aligned}\det W_1&=\det V'\prod_{i=2,N}(z-\lambda_i)(\bar{z}-\bar{\lambda}_i),\\ \det W_2&=\det V\prod_{i=2,N}(z-\lambda_i)(\bar{z}-\bar{\lambda}_i),\end{aligned} \tag{A4}$$

the ratio of the determinants $\det W_1/\det W_2$ is simply $(V^{-1})_{11}$. In the above, V' is the minor of V left after crossing out the first row and first column. In this way we obtain

$$\frac{1}{2}\partial_\epsilon\ln\det[\cdots]=\frac{1}{\epsilon}\frac{1}{1+u^2\dfrac{1}{V_{11}(V^{-1})_{11}}}. \tag{A5}$$

The determinant corresponding to the second trace in Eq. (A1), with $(z-\mathcal{M})$ and $(\bar{z}-\mathcal{M}^{\dagger})$ interchanged, is obtained by substituting $V\leftrightarrow V^{-1}$ and hence is given by exactly the same formula.

Therefore,

$$\frac{1}{4}\langle \partial_\epsilon \ln\det[\cdots]\partial_\epsilon \ln\det[\cdots]\rangle = \left\langle \frac{1}{\epsilon^2}\frac{1}{\left(1+u^2\dfrac{1}{V_{11}(V^{-1})_{11}}\right)^2}\right\rangle \overset{\epsilon\to 0}{=} \langle \pi\delta^{(2)}(z-\lambda_1)V_{11}(V^{-1})_{11}\rangle, \tag{A6}$$

where we used the representation for the complex Dirac delta $\pi\delta^{(2)}(z)=\lim_{\epsilon\to 0}\epsilon^2/(\epsilon^2+|z|^2)^2$.

In this way we obtain the important formula

$$\frac{1}{4}\langle \partial_\epsilon \ln\det[\cdots]\partial_\epsilon \ln\ \det[\cdots]\rangle = \pi\left\langle \sum_i V_{ii}(V^{-1})_{ii}\delta^2(z-\lambda_i)\right\rangle, \tag{A7}$$

where we have reinstated the sum over all eigenvalues. It remains to show that $V_{ii}(V^{-1})_{ii}=\langle R_i|R_i\rangle\langle L_i|L_i\rangle$. Using any orthonormal basis $\{e_i\}$ and the decomposition (2), we see that the linear transformation U satisfying

$$U^{-1}\mathcal{M}U=\sum_k \lambda_k|e_k\rangle\langle e_k| \tag{A8}$$

can be written explicitly as

$$U=\sum_k |R_k\rangle\langle e_k|, \qquad V^{-1}=U^{\dagger}U=\sum_{k,n}|e_k\rangle\langle R_k|R_n\rangle\langle e_n|. \tag{A9}$$

From the last equation we infer

$$V_{ii}^{-1}\equiv\langle e_i|V^{-1}|e_i\rangle=\langle R_i|R_i\rangle. \tag{A10}$$

Similar reasoning leads to $V_{ii}=\langle L_i|L_i\rangle$.

This yields

$$\frac{1}{4}\langle \partial_\epsilon \ln\det[\cdots]\partial_\epsilon \ln\det[\cdots]\rangle = \pi\left\langle \sum_i \langle L_i|L_i\rangle\langle R_i|R_i\rangle\delta^{(2)}(z-\lambda_i)\right\rangle \equiv \pi N O(z), \tag{A11}$$

which completes the proof of Eq. (A1).

We would like to stress that until now we have not used the large-N arguments in deriving this formula. Therefore, Eq. (A1) is more general than the relation (7), and could be used as a starting point for a systematic study of finite-size effects or microscopic limit.

To complete the proof of relation (7), we observe that, in the large-N limit, we could use the factorization theorem, such that the left-hand side of Eq. (A1) splits into the product of averages

$$\left\langle \mathrm{Tr}\frac{\epsilon}{|z-\mathcal{M}|^2+\epsilon^2}\mathrm{Tr}\frac{\epsilon}{|z-\mathcal{M}|^2+\epsilon^2}\right\rangle = \left\langle \mathrm{Tr}\frac{\epsilon}{|z-\mathcal{M}|^2+\epsilon^2}\right\rangle\left\langle \mathrm{Tr}\frac{\epsilon}{|z-\mathcal{M}|^2+\epsilon^2}\right\rangle. \tag{A12}$$

Below we will give some heuristic arguments for the validity of factorization in this context. However, a full proof would require a careful analysis of the interplay between the two limits $N\to\infty$ and $\epsilon\to 0$. We will not do this here but remark that the extensive numerical evidence for the diverse random matrix ensembles considered in this paper strongly suggests that no subtleties arise and that the factorization theorem holds in this case.

The validity of the factorization theorem can be argued as follows. Consider $w_1\equiv z$ and $w_2\equiv\bar{z}$ in Eq. (A12) as independent complex variables. When both of them are very large, one can prove the factorization diagrammatically and no singularities would appear (the matrices in the denominators are then invertible). Thus both sides of Eq. (A12) are equal and holomorphic in w_1 and w_2 near infinity. For fixed ϵ (and this is what we need), we see that the matrices in the denominators are invertible for some (ϵ-dependent) neighborhood of the diagonal $w_1=\overline{w_2}$. Therefore, no singularities are encountered and both sides of Eq. (A12) are holomorphic in this neighborhood. By the identity theorem since they are equal near infinity and holomorphic, they are equal everywhere. Therefore, the factorization Eq. (A12) should hold for any z.

We note that in all cases where singularities for coinciding points do appear in the $1/N$ corrections to a factorization of the form $\langle G(z)G(z)\rangle=\langle G(z)\rangle\langle G(z)\rangle$, the singularities are associated with inverting a singular matrix, and occur for z lying in (or at the edge of) the spectrum. In our case the ϵ^2 regularizes such a possible singularity.

From the definitions (8) and (9), the off-diagonal Green's functions have the following expression:

$$\mathcal{G}_{qq}^{-}=\left\langle -i\lim_{\epsilon\to 0}\frac{1}{N}\mathrm{Tr}\frac{\epsilon}{(z-\mathcal{M})(\bar{z}-\mathcal{M}^{\dagger})+\epsilon^2}\right\rangle, \qquad \mathcal{G}_{qq}^{-}=\left\langle -i\lim_{\epsilon\to 0}\frac{1}{N}\mathrm{Tr}\frac{\epsilon}{(\bar{z}-\mathcal{M}^{\dagger})(z-\mathcal{M})+\epsilon^2}\right\rangle \tag{A13}$$

Hence

$$\frac{\pi}{N}O(z)=-\mathcal{G}_{qq}^{-}\mathcal{G}_{qq}^{-}, \tag{A14}$$

which is just Eq. (7).

APPENDIX B: GENERALIZED BLUE'S FUNCTION

The generalized Blue's function [4,5] is a 2×2 matrix-valued function defined by

$$\mathcal{B}[\mathcal{G}(\mathcal{Z})]=\mathcal{Z}=\begin{pmatrix} z & i\epsilon \\ i\epsilon & \overline{z} \end{pmatrix}, \tag{B1}$$

where $\mathcal{G}$ was defined in Sec. II and ϵ will be eventually set to zero. This is equivalent to the definition in terms of the self-energy matrix,

$$\mathcal{B}(\mathcal{G})=\Sigma+\mathcal{G}^{-1}, \tag{B2}$$

where Σ is a 2×2 self-energy matrix expressed as a function of a matrix-valued Green's function. The addition law for the generalized Blue's functions reads

$$\mathcal{Z}=\mathcal{B}_1(\mathcal{G})+\mathcal{B}_2(\mathcal{G})-\mathcal{G}^{-1}, \tag{B3}$$

in analogy to the original construction by Zee [17] for Hermitian matrices.

1. Ginibre's ensemble

Ginibre's ensemble [8] could be viewed as a sum of Hermitian and anti-Hermitian Gaussian ensembles, with the original width suppressed by $\sqrt{2}$ in relation to the original width of the complex Gaussian ensemble. The generalized Blue's function for the Hermitian part is simply [4]

$$\mathcal{B}_R(\mathcal{A})=\tfrac{1}{2}\mathcal{A}+\mathcal{A}^{-1}. \tag{B4}$$

The generalized Blue's function for the anti-Hermitian part is [4]

$$\mathcal{B}_{iR}(\mathcal{A})=\tfrac{1}{2}\tilde{\mathcal{A}}+\mathcal{A}^{-1}, \tag{B5}$$

where we used the notation

$$\tilde{\mathcal{A}}=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\mathcal{A}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B6}$$

The factor $\frac{1}{2}$ comes from the normalization of the width, and the extra signs reflect on the anti-Hermitian correlations of the matrix elements. The addition law now reads (in the matrix form)

$$\mathcal{Z}=\mathcal{G}^{-1}+\tfrac{1}{2}[\mathcal{G}+\tilde{\mathcal{G}}]. \tag{B7}$$

The nontrivial (nonholomorphic solution) reads

$$\mathcal{G}=\begin{pmatrix} \overline{z} & \sqrt{|z|^2-1} \\ \sqrt{|z|^2-1} & z \end{pmatrix}. \tag{B8}$$

The domain of eigenvalues is determined by the condition $\mathcal{G}_{q\bar{q}}=0$, for which the block structure decouples leading to holomorphic and antiholomorphic copies. For this ensemble, this is simply a circle $|z|^2=1$. Inside the circle, $\mathcal{G}_{qq}(z,\overline{z})=\overline{z}$ (upper left corner of $\mathcal{G}$). The constant density of eigenvalues follows from Gauss' law (10). Outside the circle, the second (holomorphic) solution of Eq. (B7) is valid, giving $G(z)=1/z$. This reproduces the salient features of Ginibre's ensemble.

A straightforward generalization [4] using the measure [9,10]

$$P(\mathcal{M})d\mathcal{M}\sim\exp\left(-\frac{N}{1-\tau^2}\,\mathrm{Tr}\,(\mathcal{M}\mathcal{M}^{\dagger}-\tau\,\mathrm{Re}\,\mathcal{M}\mathcal{M})\right)d\mathcal{M} \tag{B9}$$

leads to the results for the elliptic ensemble with

$$\mathcal{G}_{qq}=\frac{\overline{z}-\tau z}{1-\tau^2},$$

$$\mathcal{G}_{\bar{q}\bar{q}}=\overline{\mathcal{G}}_{qq}, \tag{B10}$$

$$\mathcal{G}_{q\bar{q}}\mathcal{G}_{q\bar{q}}=\frac{(1+\tau^2)|z|^2-2\tau\,\mathrm{Re}\,z^2-(1-\tau^2)^2}{(1-\tau^2)^2}.$$

Note that the measure (B9) leads to nonvanishing cumulants $\langle\mathcal{M}_{ab}\overline{\mathcal{M}}_{ab}\rangle=1/N$ and $\langle\mathcal{M}_{ab}\mathcal{M}_{ba}\rangle=\tau/N$. In particular, $\tau=-1$ corresponds to anti-Hermitian matrices, explaining the flips of the signs in the tilted variables above.

2. Two-level deterministic Hamiltonian plus Ginibre's ensemble

Since the deterministic Hermitian Green's function for the two-level Hamiltonian is

$$G_D(z)=\frac{1}{2}\left(\frac{1}{z-a}+\frac{1}{z+a}\right), \tag{B11}$$

the corresponding generalized Green's function for this Hamiltonian reads

$$\mathcal{G}_D(\mathcal{Z})=\tfrac{1}{2}[(\mathcal{Z}-a\mathbf{1}_2)^{-1}+(\mathcal{Z}+a\mathbf{1}_2)^{-1}] \tag{B12}$$

with $\mathbf{1}_2$ denoting the two-dimensional identity matrix. Substituting in Eq. (B12) $\mathcal{Z}\to\mathcal{B}_D(\mathcal{G})$, we obtain

$$\mathcal{G}=\tfrac{1}{2}\{[\mathcal{B}_D(\mathcal{G})-a\mathbf{1}_2]^{-1}+[\mathcal{B}_D(\mathcal{G})+a\mathbf{1}_2]^{-1}\}. \tag{B13}$$

The addition law for the generalized Blue's functions reads

$$\mathcal{B}(\mathcal{A})=\mathcal{B}_D(\mathcal{A})+\mathcal{B}_G(\mathcal{A})-\mathcal{A}^{-1}, \tag{B14}$$

where the generalized Blue's function for Ginibre's ensemble was constructed above, i.e., $\mathcal{B}_G(\mathcal{A})=\mathcal{A}^{-1}+1/2(\mathcal{A}+\tilde{\mathcal{A}})$. Substituting in Eq. (B14) $\mathcal{A}\to\mathcal{G}(\mathcal{Z})$, we infer the relation

$$\mathcal{Z}=\mathcal{B}_D(\mathcal{G})+\tfrac{1}{2}(\mathcal{G}+\tilde{\mathcal{G}}). \tag{B15}$$

From Eqs. (B13) and (B15) we get the final equation,

$$\mathcal{G}=\tfrac{1}{2}\{(\mathcal{Z}-\tfrac{1}{2}(\mathcal{G}+\tilde{\mathcal{G}})-a\mathbf{1}_2]^{-1}+[\mathcal{Z}-\tfrac{1}{2}(\mathcal{G}+\tilde{\mathcal{G}})+a\mathbf{1}_2]^{-1}\}. \tag{B16}$$

Solving this matrix equation, we arrive at

$$\mathcal{G}_{qq}\mathcal{G}_{q\bar{q}}=|z|^2+a^2-\tfrac{1}{2}(1+\sqrt{1+4a^2(z+\bar{z})^2}),$$

$$\mathcal{G}_{qq}=\bar{z}-\frac{2a^2(z+\bar{z})}{1+\sqrt{1+4a^2(z+\bar{z})^2}}, \tag{B17}$$

$$\mathcal{G}_{\bar{q}\bar{q}}=\bar{\mathcal{G}}_{qq}.$$

3. Open chaotic scattering

Since the addition law for open chaotic scattering was formulated by us in previous publications [4], we will be brief and refer to the original papers for details. The addition law reads

$$\mathcal{Z}=m(1-\Gamma\mathcal{G})^{-1}\Gamma+\mathcal{G}+\mathcal{G}^{-1}, \tag{B18}$$

where $m=M/N$ and $\Gamma=\mathrm{diag}(-ig,ig)$. Solution of the matrix equation (B18) leads to

$$\mathcal{G}_{qq}=\frac{x}{2}+\frac{i}{2}\left[\frac{1}{g}+\frac{m}{y}+\frac{g}{1-gy}\right],$$

$$\mathcal{G}_{qq}\mathcal{G}_{q\bar{q}}=\frac{x^2}{4}+\frac{1}{4}\left(\frac{g}{1-gy}+\frac{m}{y}+\frac{1}{g}\right)^2-\frac{1}{1-gy}, \tag{B19}$$

$$\mathcal{G}_{\bar{q}\bar{q}}=\bar{\mathcal{G}}_{qq},$$

where $z=x+iy$. Gauss' law leads to the spectral density [12]

$$4\pi\nu(x,y)=\mathrm{div}\vec{E}=1+\frac{m}{y^2}-\frac{g^2}{(1-gy)^2}, \tag{B20}$$

where $E_x=2\mathrm{Re}\,\mathcal{G}_{qq}$ and $E_y=-2\mathrm{Im}\mathcal{G}_{qq}$. For completeness we mention that condition $\mathcal{G}_{q\bar{q}}=0$ reproduces the results by Haake, Sommers, and co-workers [12] for the boundary of eigenvalues in open chaotic scattering.

[1] Y. V. Fyodorov and H.-J. Sommers, J. Math. Phys. **38**, 1918 (1997); Y. V. Fyodorov, B. A. Khoruzhenko, and H.-J. Sommers, Phys. Lett. A **226**, 46 (1997).

[2] K. B. Efetov, Phys. Rev. Lett. **79**, 491 (1997); Phys. Rev. B **56**, 9630 (1997).

[3] N. Hatano and D. R. Nelson, Phys. Rev. Lett. **77**, 370 (1996).

[4] R. A. Janik, M. A. Nowak, G. Papp, and I. Zahed, Nucl. Phys. B **501**, 603 (1997); R. A. Janik, M. A. Nowak, G. Papp, J. Wambach, and I. Zahed, Phys. Rev. E **55**, 4100 (1997).

[5] J. Feinberg and A. Zee, Nucl. Phys. B **501**, 643 (1997); **504**, 579 (1997).

[6] J. T. Chalker and B. Mehlig, Phys. Rev. Lett. **81**, 3367 (1998).

[7] M. Müller, F.-M. Dittes, W. Iskra, and I. Rotter, Phys. Rev. E **52**, 5961 (1995).

[8] J. Ginibre, J. Math. Phys. **6**, 440 (1965).

[9] V. L. Girko, *Spectral Theory of Random Matrices* (Nauka, Moscow, 1988) (in Russian), and references therein.

[10] H.-J. Sommers, A. Crisanti, H. Sompolinsky, and Y. Stein, Phys. Rev. Lett. **60**, 1895 (1988).

[11] C. Mahaux and H. A. Weidenmüller, *Shell Model Approach to Nuclear Reactions* (North Holland, Amsterdam, 1969).

[12] F. Haake *et al.*, Z. Phys. B **88**, 359 (1992); N. Lehmann, D. Saher, V. V. Sokolov, and H.-J. Sommers, Nucl. Phys. A **582**, 223 (1995).

[13] V. V. Sokolov and V. G. Zelevinsky, Phys. Lett. B **202**, 101 (1988); Nucl. Phys. A **504**, 562 (1989).

[14] R. A. Janik, M. A. Nowak, G. Papp, and I. Zahed, in *New Developments in Quantum Field Theory*, edited by P. Damgaard and J. Jurkiewicz, Vol. 366 of *NATO Advanced Study Institute Series B: Physics* (Plenum Press, New York, 1998), p. 297.

[15] R. A. Janik, M. A. Nowak, G. Papp, and I. Zahed, Acta Phys. Pol. B **28**, 2949 (1997).

[16] J. T. Chalker and Z. Jane Wang, Phys. Rev. Lett. **79**, 1797 (1997).

[17] A. Zee, Nucl. Phys. B **474**, 726 (1996).